Explicit expressions for the electric and magnetic fields of a moving magnetic dipole

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Explicit expressions for the electric and magnetic fields of an arbitrarily moving particle possessing a constant magnetic moment are derived from retarded integrals representing the solution of Maxwell's equations for electric and magnetic fields of a magnetized source. These expressions exhibit explicitly the useful separation of the fields into their $1/R$, $1/R^2$, and $1/R^3$ parts. The total power radiated by this magnetic dipole is then calculated when the velocity, acceleration, and the derivative of acceleration are parallel. The low velocity limit of this power and the conservation of energy are used to derive a nonlinear damping force acting on a nonrelativistic magnetic dipole. [S1063-651X(98)09709-8]

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I. INTRODUCTION

It is well known that a dipole *at rest* (at the source point $\mathbf{x}'=\mathbf{x}_0$) with a constant magnetic moment $\boldsymbol{\mu}$ yields the magnetostatic field: $\mathbf{B}=[3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}]/R^3$, where $R=|\mathbf{x}-\mathbf{x}_0|$ is the magnitude of $\mathbf{R}=(\mathbf{x}-\mathbf{x}_0)$, with $\mathbf{x}$ being the field point and $\mathbf{n}=\mathbf{R}/R$. The natural question then arises: How is this magnetostatic field modified when the dipole is set in arbitrary motion? This basic question of classical electrodynamics was treated in the 1960s by several authors [1]. However, it is somewhat surprising to note that none of them derived *explicit* expressions for the fields of an arbitrarily moving particle with a constant magnetic moment in terms of conventional parameters $(\mathbf{n},\boldsymbol{\beta},\dot{\boldsymbol{\beta}},...)$ and exhibiting the useful separation of the fields into their $1/R$, $1/R^2$, and $1/R^3$ parts. However, expressions for the $1/R$ part of these fields, the so-called radiation fields, have recently been derived [2,3].

In this paper general formulas for the electric and magnetic fields due to a magnetized source are derived by making the replacements $\rho=0$ and $\mathbf{J}\to c\nabla\times\mathbf{M}$ into the generalized Coulomb and Biot-Savart laws [4–6] and using certain integral relations that are proved in the Appendix. The general formulas are first applied to find the electric and magnetic fields of an oscillating magnetic dipole by recognizing that the magnetic field contains a delta term that has been ignored by the standard literature. It is shown that this delta term accounts for the interaction energy of two oscillating magnetic dipoles. In a second application, novel expressions for the electric and magnetic fields of a moving particle possessing a constant magnetic moment are derived. These fields are given in terms of their explicit $1/R$, $1/R^2$, and $1/R^3$ parts. The total power radiated by this magnetic dipole is then determined, in the simple case in which the vectors $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, and $\ddot{\boldsymbol{\beta}}$ are parallel. The low velocity limit of this radiated power is used to derive a nonlinear damping force from the conservation of energy. It is pointed out that the linearized version of this force corresponds (up to a constant) with the expression for a radiation damping force of a quantum particle with a spin magnetic moment recently derived by Smirnov [7].

The organization of this paper is as follows. In Sec. II formulas for electric and magnetic fields of a magnetized source are derived. In Sec. III these formulas are applied to finding the fields of an oscillating magnetic dipole. In Sec. IV the electric and magnetic fields of an arbitrarily moving particle with a constant magnetic moment are derived. These fields are expressed in terms of their $1/R$, $1/R^2$, and $1/R^3$ parts in Sec. V. The power radiated by the magnetic dipole is obtained in Sec. VI and used to find a radiation damping force.

II. THE FIELDS OF A MAGNETIZED SOURCE

Consider the time-dependent generalizations of the Coulomb and Biot-Savart laws as given by Jefimenko [4–6]. In Gaussian units these laws can be written as [3]

$$\mathbf{E}=\int\int\delta(u)\left(\frac{\rho\mathbf{n}}{R^2}+\frac{\dot{\rho}n}{Rc}-\frac{\dot{\mathbf{J}}}{Rc^2}\right)d^3x'\,dt', \tag{1a}$$

$$\mathbf{B}=-\int\int\delta(u)\left(\frac{\mathbf{n}\times\mathbf{J}}{R^2c}+\frac{\mathbf{n}\times\dot{\mathbf{J}}}{Rc^2}\right)d^3x'\,dt', \tag{1b}$$

where the time integration is from $-\infty$ to $+\infty$ and the spatial integration is over all space; the retardation effect is provided by the delta function $\delta(u)$ with $u=t'+R/c-t$ where $R=|\mathbf{x}-\mathbf{x}'|$ and the overdot means differentiation with respect to t'. Equations (1) can be used for deriving general expressions for the electric and magnetic fields of a magnetized source $\mathbf{M}$ in vacuum—in the context of this paper, the vector $\mathbf{M}$ is assumed to be the source of the fields $\mathbf{E}$ and $\mathbf{B}$. This means that $\mathbf{M}$ must be defined independently of the proper fields $\mathbf{E}$ and $\mathbf{B}$ (as a dipole moment density, for example).

The time derivative in the last term of Eq. (1a) can be taken out of the integral by performing an integration by parts and using the property $\partial\delta(u)/\partial t'=-\partial\delta(u)/\partial t$. Thus, one has the equivalent form of Eq. (1a):

$$\mathbf{E}=\int\int\delta(u)\left(\frac{\rho\mathbf{n}}{R^2}+\frac{\dot{\rho}\mathbf{n}}{Rc}\right)d^3x'\,dt' -\frac{\partial}{\partial t}\int\int\frac{\delta(u)\mathbf{J}}{Rc^2}\,d^3x'\,dt'. \tag{2}$$

With the formal substitutions $\rho=0$, $\mathbf{J}\rightarrow c\nabla\times\mathbf{M}$, Eq. (2) becomes

$$\mathbf{E}=-\frac{\partial}{\partial t}\int\int\frac{\delta(u)\nabla'\times\mathbf{M}}{Rc}\,d^3x'\,dt'. \tag{3}$$

The spatial derivative in this expression can be removed by recasting it into time derivatives. In the Appendix it is shown that

$$\int\int\frac{\delta(u)\nabla'\times\mathbf{M}}{Rc}\,d^3x'\,dt' = -\int\int\delta(u)\left(\frac{\mathbf{n}\times\mathbf{M}}{R^2c}+\frac{\mathbf{n}\times\dot{\mathbf{M}}}{Rc^2}\right)d^3x'\,dt'. \tag{4}$$

Equations (3) and (4) combine to yield

$$\mathbf{E}=\int\int\delta(u)\left(\frac{\mathbf{n}\times\dot{\mathbf{M}}}{R^2c}+\frac{\mathbf{n}\times\ddot{\mathbf{M}}}{Rc^2}\right)d^3x'\,dt'. \tag{5}$$

Consider now Eq. (1b) rewritten in the equivalent form

$$\mathbf{B}=-\int\int\frac{\delta(u)\mathbf{n}\times\mathbf{J}}{R^2c}\,d^3x'\,dt' - \frac{\partial}{\partial t}\int\int\frac{\delta(u)\mathbf{n}\times\mathbf{J}}{Rc^2}\,d^3x'\,dt'. \tag{6}$$

With the substitution $\mathbf{J}\rightarrow c\nabla\times\mathbf{M}$, Eq. (6) becomes

$$\mathbf{B}=-\int\int\frac{\delta(u)\mathbf{n}\times(\nabla'\times\mathbf{M})}{R^2}\,d^3x'\,dt' - \frac{\partial}{\partial t}\int\int\frac{\delta(u)\mathbf{n}\times(\nabla'\times\mathbf{M})}{Rc}\,d^3x'\,dt'. \tag{7}$$

In the Appendix it is shown that

$$\int\int\frac{\delta(u)\mathbf{n}\times(\nabla'\times\mathbf{M})}{R^2}\,d^3x'\,dt' = -\int\int\delta(u)\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M})-\mathbf{M}}{R^3}+\frac{\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{M}})-\dot{\mathbf{M}}}{R^2c}\right)\times d^3x'\,dt' - \frac{8\pi\mathbf{M}}{3}, \tag{8}$$

$$\int\int\frac{\delta(u)\mathbf{n}\times(\nabla'\times\mathbf{M})}{Rc}\,d^3x'\,dt' = -\int\int\delta(u)\left(\frac{2\mathbf{n}(\mathbf{n}\cdot\mathbf{M})}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{M}})}{Rc^2}\right)d^3x'\,dt' \tag{9}$$

and therefore Eqs. (7)–(9) combine to give the expression

$$\mathbf{B}=\int\int\delta(u)\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{M})-\mathbf{M}}{R^3}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{M}})-\dot{\mathbf{M}}}{R^2c}+\frac{\mathbf{n}\times(\mathbf{n}\times\ddot{\mathbf{M}})}{Rc^2}\right)d^3x'\,dt'+\frac{8\pi\mathbf{M}}{3}. \tag{10}$$

Equations (5) and (10) constitute the solution of Maxwell's equations

$$\nabla\cdot\mathbf{E}=0, \tag{11a}$$

$$\nabla\cdot\mathbf{B}=0, \tag{11b}$$

$$\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=0, \tag{11c}$$

$$\nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=4\pi\nabla\times\mathbf{M}, \tag{11d}$$

satisfying the conditions that the fields **E** and **B** (and their derivatives) vanish at infinity and the source **M** is confined to a finite region of space. It should be noted that the term $(8\pi/3)\mathbf{M}$ in Eq. (10) is a function evaluated at the field point and the present time. Without the presence of this ''contact'' term, Eqs. (5) and (10) do not strictly satisfy Maxwell's equations. Equation (10) may be interpreted then as follows. The field **B** is formed by two terms: The integral term represents the value of **B** outside the magnetized source while the contact term $(8\pi/3)\mathbf{M}$ represents its value inside the source. A remarkable property of Eqs. (5) and (10) is that they do not involve spatial derivatives of the vector **M**, which in most cases simplifies considerably the calculation of the fields. It should be also mentioned that Eqs. (5) and (10) (without the contact term) were previously derived in Ref. [2].

III. THE OSCILLATING MAGNETIC DIPOLE

Consider a particle *at rest* (at the point $\mathbf{x}_0$) with a magnetic moment oscillating in time $\boldsymbol{\mu}=\mu(t)\mathbf{e}$, where $\mathbf{e}$ is its direction. The associated magnetization vector is given by $\mathbf{M}(\mathbf{x},t)=\mathbf{e}\mu(t)\delta(\mathbf{x}-\mathbf{x}_0)$. With this source the integration of Eqs. (5) and (10) yields

$$\mathbf{E}=\frac{\dot{\mu}(t')\mathbf{n}\times\mathbf{e}}{R^2c}+\frac{\ddot{\mu}(t')\mathbf{n}\times\mathbf{e}}{Rc^2}, \tag{12a}$$

$$\mathbf{B}=\frac{\mu(t')\{3\mathbf{n}(\mathbf{e}\cdot\mathbf{n})-\mathbf{e}\}}{R^3}+\frac{\dot{\mu}(t')\{3\mathbf{n}(\mathbf{e}\cdot\mathbf{n})-\mathbf{e}\}}{R^2c}+\frac{\ddot{\mu}(t')\mathbf{n}\times(\mathbf{n}\times\mathbf{e})}{Rc^2}+\frac{8\pi\mu(t)\mathbf{e}}{3}\delta(\mathbf{x}-\mathbf{x}_0), \tag{12b}$$

where now $\mathbf{n}=(\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$ and the overdot means differentiation with respect to $t'=t-R/c$ with $R=|\mathbf{x}-\mathbf{x}_0|$. Equations (12) (without the delta term) are the well-known fields of an oscillating magnetic dipole.

The novelty in Eq. (12b) is the presence of the contact term $(8\pi/3)\mu(t)\mathbf{e}\delta(\mathbf{x}-\mathbf{x}_0)$, which represents the magnetic field within the dipole. It is somewhat surprising to find that this delta term is not usually mentioned in the standard literature—in the static regime, however, the analogous delta term is well known [8]; it is precisely the term that accounts for hyperfine splitting in the ground state of hydrogen [9,10]. However, the delta term is necessary here for the consistency of Eqs. (12) with the Maxwell equations. Moreover, this delta term accounts for the interaction energy of two oscillating magnetic dipoles. Indeed, the energy of an oscillating magnetic dipole $\boldsymbol{\mu}=\mu(t)\mathbf{e}$ in the presence of a magnetic field $\mathbf{B}$ is given by $U=-\mu(t)\mathbf{e}\cdot\mathbf{B}$. In particular, the energy of an oscillating magnetic dipole $\boldsymbol{\mu}_1=\mu_1(t)\mathbf{e}_1$ in the presence of the magnetic field $\mathbf{B}$ of another oscillating magnetic dipole $\boldsymbol{\mu}_2=\mu_2(t)\mathbf{e}_2$ is

$$U=-\frac{\mu_1(t')\mu_2(t)\{3(\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{n})-\mathbf{e}_1\cdot\mathbf{e}_2\}}{R^3} -\frac{\dot{\mu}_1(t')\mu_2(t)\{3(\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{n})-\mathbf{e}_1\cdot\mathbf{e}_2\}}{R^2c} -\frac{\ddot{\mu}_1(t')\mu_2(t)\{(\mathbf{e}_1\cdot\mathbf{n})(\mathbf{e}_2\cdot\mathbf{n})-\mathbf{e}_1\cdot\mathbf{e}_2\}}{Rc^2} -\frac{8\pi}{3}\mu_1(t)\mu_2(t)\delta(\mathbf{x}-\mathbf{x}_0)(\mathbf{e}_1-\mathbf{e}_2), \tag{13}$$

where $\mathbf{x}-\mathbf{x}_0$ is the separation of the dipoles. When the dipoles are separated the delta term in Eq. (13) can be ignored. However, when the dipoles are at the *same* place the delta-term contribution accounts for the interaction energy.

IV. A MOVING PARTICLE POSSESSING A CONSTANT MAGNETIC MOMENT

The problem of finding the fields of a point dipole in arbitrary motion is somewhat different from that of computing the fields of a point charge in arbitrary motion [2]. In the case of a moving charge e one solves the usual Maxwell's equations with the sources $\rho(\mathbf{x},t)=e\,\delta\{\mathbf{x}-\mathbf{r}(t)\}$ and $\mathbf{J}(\mathbf{x},t)=e\mathbf{v}(t)\,\delta\{\mathbf{x}-\mathbf{r}(t)\}$. By analogy one might think that for a moving dipole possessing a constant magnetic moment $\boldsymbol{\mu}$ the problem consists in solving Maxwell's equations (11) with the source $\mathbf{M}(\mathbf{x},t)=\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{r}(t)\}$. However, there is a subtle difference between the two cases. The electric charge is conserved and Lorentz invariant and thereby it is necessarily independent of motion. Nevertheless, the magnetic dipole moment is not a Lorentz invariant. However, in the model assumed in this paper the point dipole is observed in a frame where there is only magnetization and it is given by $\mathbf{M}(x,t)=\boldsymbol{\mu}\delta(\mathbf{x}-\mathbf{r}(t))$.

Equation (5) can be written in the equivalent form

$$\mathbf{E}=\frac{1}{c}\frac{\partial}{\partial t}\int\int\delta(t'+R/c-t)\left(\frac{\mathbf{n}\times\mathbf{M}}{R^2}\right)d^3x'\,dt' +\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int\int\delta(t'+R/c-t)\left(\frac{\mathbf{n}\times\mathbf{M}}{R}\right)d^3x'\,dt'. \tag{14}$$

With the magnetization $\mathbf{M}(\mathbf{x},t)=\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{r}(t)\}$, the volume integrals in Eq. (14) can be done immediately, yielding

$$\mathbf{E}=-\boldsymbol{\mu}\times\frac{1}{c}\frac{d}{dt}\int\delta(t'+R(t')/c-t)\left(\frac{\mathbf{n}(t')}{R(t')^2}\right)dt' -\boldsymbol{\mu}\frac{1}{c^2}\frac{d^2}{dt^2}\int\delta(t'+R(t')/c-t)\left(\frac{\mathbf{n}(t')}{R(t')}\right)dt', \tag{15}$$

where now $R(t')=|\mathbf{x}-\mathbf{r}(t')|$ and $\mathbf{n}(t')=[\mathbf{x}-\mathbf{r}(t')]/R(t')$. Using the formula

$$\int\delta(f(t')-a)\mathbf{g}(t')dt'=\left.\frac{\mathbf{g}(t')}{|df/dt'|}\right|_{f(t')=a}, \tag{16}$$

with $f(t')=t'+R(t')/c$, $a=t$, and $df/dt'=1-\mathbf{n}\cdot\boldsymbol{\beta}$, the time integrals in Eq. (15) can be performed, yielding

$$\mathbf{E}=-\boldsymbol{\mu}\times\frac{1}{c}\frac{d}{dt}\left[\frac{\mathbf{n}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right]_{\rm ret} -\boldsymbol{\mu}\frac{1}{c^2}\frac{d^2}{dt^2}\left[\frac{\mathbf{n}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right]_{\rm ret}. \tag{17}$$

This equation can also be written in the convenient form

$$\mathbf{E}=\left[\frac{1}{c}\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{\mathbf{n}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\right]_{\rm ret}\times\boldsymbol{\mu}, \tag{18}$$

on the understanding that $[dF/dt]_{\rm ret}$ means $dF(t')/dt$ and not $dF(t')/dt'$, that is, the "ret" outside the square brackets applies to the arguments of the functions inside and not to the variable of differentiation [11].

By a similar procedure, when $\mathbf{M}(\mathbf{x},t)=\boldsymbol{\mu}\delta(\mathbf{x}-\mathbf{r}(t))$ is inserted into Eq. (10) and the integrations over the resulting expression are performed one obtains the magnetic field of an arbitrarily moving magnetic dipole:

$$\mathbf{B}=\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})}+\frac{1}{c}\frac{d}{dt}\left(\frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right) +\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\right]_{\rm ret}+\frac{8\pi}{3}\boldsymbol{\mu}\delta(\mathbf{x}-\mathbf{r}(t)). \tag{19}$$

A similar expression but without the delta term was obtained by Monaghan in Ref. [1]. In the derivation of Eq. (19), however, the assumption that $\boldsymbol{\mu}$ is a constant vector has not yet been used. This means that Eq. (19) is valid even in the case that $\boldsymbol{\mu}$ is a function of time. However, by performing some derivatives in Eq. (19) and making $d\boldsymbol{\mu}/dt=0$ one obtains

$$\mathbf{B}=\Bigg[\frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})}+\frac{3}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\left\{\frac{1}{c}\frac{d\mathbf{n}}{dt}(\mathbf{n}\cdot\boldsymbol{\mu})+\mathbf{n}\left(\frac{1}{c}\frac{d\mathbf{n}}{dt}\cdot\boldsymbol{\mu}\right)\right\}+\{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}\}\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)$$
$$+\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})\}\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)+\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\left\{\frac{2}{c^2}\frac{d^2\mathbf{n}}{dt^2}\times(\mathbf{n}\times\boldsymbol{\mu})+\boldsymbol{\mu}\times\left(\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\times\mathbf{n}\right)+\frac{2}{c}\frac{d\mathbf{n}}{dt}\times\left(\frac{1}{c}\frac{d\mathbf{n}}{dt}\cdot\boldsymbol{\mu}\right)\right\}$$
$$+\left\{\frac{4}{c}\frac{d\mathbf{n}}{dt}\times(\mathbf{n}\times\boldsymbol{\mu})+2\boldsymbol{\mu}\times\left(\frac{1}{c}\frac{d\mathbf{n}}{dt}\times\mathbf{n}\right)\right\}\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\Bigg]_{\rm ret}+\frac{8\pi}{3}\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{r}(t)\}. \tag{20}$$

This is an expression for the magnetic field of a moving dipole with constant magnetic moment.

V. THE EXPLICIT $1/R^3$, $1/R^2$, AND $1/R$ PARTS OF THE FIELDS

Although Eqs. (18) and (19) are relatively simple, they do not exhibit explicitly the useful separation of the fields into their $1/R$, $1/R^2$, and $1/R^3$ parts. Such a separation of the fields, however, can be accomplished by performing all the specified time derivatives in Eqs. (18) and (20). This task, although straightforward, is extremely laborious. It involves long and complicated vector manipulations and the full expressions obtained for the fields turn out to be very lengthy.

It is convenient to begin with Eq. (18) rewritten as

$$\mathbf{E}=\Bigg[\mathbf{n}\left\{\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\right\}+\frac{1}{c}\frac{d\mathbf{n}}{dt}\left\{\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}+\frac{2}{c}\frac{d}{dt}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\right\}$$
$$+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)\Bigg]_{\rm ret}\times\boldsymbol{\mu}. \tag{21}$$

By using the result [3]

$$\frac{1}{c}\frac{d}{dt}(R^i(1-\mathbf{n}\cdot\boldsymbol{\beta})^j)=jR^i(1-\mathbf{n}\cdot\boldsymbol{\beta})^{j-2}\left(-\frac{\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{c}+\frac{(\mathbf{n}\times\boldsymbol{\beta})^2}{R}\right),$$
$$-iR^{i-1}(1-\mathbf{n}\cdot\boldsymbol{\beta})^{j-1}(\mathbf{n}\cdot\boldsymbol{\beta})\quad (i,j\ \text{integers}), \tag{22}$$

($\dot{\boldsymbol{\beta}}=d\boldsymbol{\beta}/dt'$) one obtains

$$\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)=\frac{\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3c}+\frac{\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}, \tag{23a}$$

$$\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)=\frac{\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3c}+\frac{\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}+\frac{\mathbf{n}\cdot\boldsymbol{\beta}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}, \tag{23b}$$

$$\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right)=\frac{3\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c}-\frac{2(\mathbf{n}\times\boldsymbol{\beta})^2-\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}, \tag{23c}$$

$$\frac{1}{c}\frac{d}{dt}\left(\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right)=\frac{3\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c}-\frac{(\mathbf{n}\times\boldsymbol{\beta})^2-2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}. \tag{23d}$$

The last two expressions are used to derive the following result:

$$\frac{1}{c^2}\frac{d^2}{dt^2}\left(\frac{1}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right)=\frac{3(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c^2}+\frac{\mathbf{n}\cdot\ddot{\boldsymbol{\beta}}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c^2}+\frac{2\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c}$$
$$+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}-2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}+\mathbf{n}\cdot\dot{\boldsymbol{\beta}}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c}+\frac{\boldsymbol{\beta}\cdot(\mathbf{n}-\beta)\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}-\frac{(\mathbf{n}\times\boldsymbol{\beta})^2}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}, \tag{23e}$$

where $\ddot{\boldsymbol{\beta}}=d^2\boldsymbol{\beta}/dt'^2$. The time derivatives of the unit vector $\mathbf{n}$ are given by

$$\frac{1}{c}\frac{d\mathbf{n}}{dt}=\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}, \tag{23f}$$

$$\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}=\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3c}+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})(1-\beta^2)}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}-\frac{\mathbf{n}(\mathbf{n}\times\boldsymbol{\beta})^2}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}-\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}. \tag{23g}$$

This last expression was derived by Wang [12]. Substitution of Eqs. (23a), (23b), and (23e)–(23g) into Eq. (21) yields an expression whose terms are arranged in order of decreasing powers of $1/R$:

$$\begin{aligned}\mathbf{E}=\Bigg[&\frac{\mathbf{n}[\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}]}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})+1-\beta^2\}-\mathbf{n}(\mathbf{n}\times\boldsymbol{\beta})^2}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}+\frac{2\mathbf{n}\{\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})\}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\\
&+\frac{2\mathbf{n}\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c}+\frac{\mathbf{n}\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}-2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}+\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\}+\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}+2\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c}\\
&+\frac{\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3c}+\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c^2}+\frac{\mathbf{n}(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c^2}\Bigg]_{\rm ret}\times\boldsymbol{\mu}.\end{aligned} \tag{24a}$$

By a similar procedure, the use of Eqs. (23a), (23b), (23e)–(23g) into Eq. (20) yields

$$\begin{aligned}\mathbf{B}=\Bigg[&\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}\{\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})\}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^5}-\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})(\mathbf{n}\times\boldsymbol{\beta})^2}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}\\
&+\frac{\boldsymbol{\mu}\times(\mathbf{n}\times\boldsymbol{\beta})[1+2(\mathbf{n}\cdot\boldsymbol{\beta})-3\beta^2]+2\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\}\times(\mathbf{n}\times\boldsymbol{\mu})[1+2(\mathbf{n}\cdot\boldsymbol{\beta})-3\beta^2]}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^4}\\
&+\frac{\{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}\}\{\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})\}-2\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})(\mathbf{n}\times\boldsymbol{\beta})^2+2\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\cdot\boldsymbol{\mu}\}-2\boldsymbol{\mu}(\mathbf{n}\times\boldsymbol{\beta})^2}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\\
&+\frac{3\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})(\mathbf{n}\cdot\boldsymbol{\mu})+3\mathbf{n}\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\cdot\boldsymbol{\mu}\}+\{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}\}(\mathbf{n}\cdot\boldsymbol{\beta})-2\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\}\times(\mathbf{n}\times\boldsymbol{\mu})+\boldsymbol{\mu}\times(\mathbf{n}\times\boldsymbol{\beta})}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\\
&+\frac{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}}{R^3(1-\mathbf{n}\cdot\boldsymbol{\beta})}+\frac{2\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})\{2\boldsymbol{\beta}\cdot(\mathbf{n}-\boldsymbol{\beta})-(\mathbf{n}\times\boldsymbol{\beta})^2\}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c}+\frac{\boldsymbol{\mu}\times[\mathbf{n}\times\{\dot{\boldsymbol{\beta}}+\mathbf{n}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\}]}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c}\\
&+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\cdot\dot{\boldsymbol{\beta}}-2\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}+\mathbf{n}\cdot\dot{\boldsymbol{\beta}}\}+[2\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}]\times(\mathbf{n}\times\boldsymbol{\mu})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c}\\
&+\frac{4\{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})\}\times(\mathbf{n}\times\boldsymbol{\mu})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})+2\boldsymbol{\mu}\times(\mathbf{n}\times\boldsymbol{\beta})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c}+\frac{\{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}\}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3c}+\frac{3\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c^2}\\
&+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\mu})(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c^2}\Bigg]_{\rm ret}+\frac{8\pi}{3}\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{r}(t)\}.\end{aligned} \tag{24b}$$

The complicated expressions in Eqs. (24) represent the electric and magnetic fields of an arbitrarily moving particle with a constant magnetic moment. It is now possible to answer the question in the Introduction: How is the magnetostatic field of a magnetic dipole modified when it is set in arbitrary motion? Answer: An arbitrarily moving magnetic dipole modifies strongly its magnetostatic field in accordance with Eqs. (24). However, a detailed interpretation of these equations is a very complicated task. But there are some points that are relevant for an interpretation of these equations:

(1) *Near, intermediate, and far fields.* As may be seen in Eqs. (24), the fields separate naturally into three parts: The *near* fields $\mathbf{E}_{\rm near}$ and $\mathbf{B}_{\rm near}$, which vary as $1/R^3$ and depend on the velocity and magnetic moment; the *intermediate* fields $\mathbf{E}_{\rm int}$ and $\mathbf{B}_{\rm int}$, which vary as $1/R^2$ and depend on the velocity, acceleration, and the magnetic moment; and the *far* fields $\mathbf{E}_{\rm far}$ and $\mathbf{B}_{\rm far}$, which vary as $1/R$ and depend on the velocity, acceleration, time derivative of acceleration, and the magnetic moment. Thus, the complete fields read $\mathbf{E}=\mathbf{E}_{\rm near}+\mathbf{E}_{\rm int}+\mathbf{E}_{\rm far}$, and $\mathbf{B}=\mathbf{B}_{\rm near}+\mathbf{B}_{\rm int}+\mathbf{B}_{\rm far}+\mathbf{B}_{\rm del}$, where $\mathbf{B}_{\rm del}$ denotes the delta-function term $(8\pi/3)\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{r}(t)\}$, which is evaluated at the field point and the present time—this term represents the magnetic field within the moving dipole; it is essential for achieving the consistency of Eqs. (24) with the Maxwell equations (11).

(2) *Static limit.* When the velocity, acceleration, and derivative of acceleration of the magnetic dipole are zero, that is, when the dipole is at absolute rest, Eq. (24a) yields $\mathbf{E}=0$ and Eq. (24b) reduces to the well known static form: $\mathbf{B}=\{3\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-\boldsymbol{\mu}\}/R^3+(8\pi/3)\boldsymbol{\mu}\delta\{\mathbf{x}-\mathbf{x}_0\}$, where $\mathbf{x}_0$ is the point where the dipole is at rest [8].

(3) *Uniform motion.* It follows from Eqs. (24) that, in contrast to the simple form of the fields of a charge moving with constant velocity, the fields of a magnetic dipole in

uniform motion ($\dot{\boldsymbol{\beta}}=\ddot{\boldsymbol{\beta}}=0$) exhibit an exceedingly complicated form. The electric field is given by the $1/R^3$ part of Eq. (24a) and the magnetic field by the $1/R^3$ part of Eq. (24b) plus the delta term, that is, $\mathbf{E}=\mathbf{E}_{\rm near}$ and $\mathbf{B}=\mathbf{B}_{\rm near}+\mathbf{B}_{\rm del}$.

(4) *Coulombian acceleration fields.* The presence of the acceleration vector in the intermediate fields is remarkable—for a moving charge the acceleration appears only in the far fields. This result is better recognized when one assumes low velocities. Indeed, if the velocity of the dipole is small compared with that of light ($\beta \ll 1$) then the intermediate fields in Eqs. (24) reduce to

$$\mathbf{E}_{\rm int}=\left[\frac{3\mathbf{n}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})-\dot{\boldsymbol{\beta}}}{R^2c}\right]_{\rm ret}\times\boldsymbol{\mu}, \tag{25a}$$

$$\mathbf{B}_{\rm int}=\left[\frac{\{6\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\mu})-2\boldsymbol{\mu}\}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})-\mathbf{n}(\boldsymbol{\mu}\cdot\dot{\boldsymbol{\beta}})-\dot{\boldsymbol{\beta}}(\mathbf{n}\cdot\boldsymbol{\mu})}{R^2c}\right]_{\rm ret}. \tag{25b}$$

These fields may be called "Coulombian acceleration fields." The point here is that these fields, though depending linearly on the acceleration, are not radiation fields since they vary as $1/R^2$.

(5) *Radiation fields.* Evidently, the far fields appearing in Eqs. (24),

$$\mathbf{E}_{\rm rad}=\left[\frac{3\mathbf{n}\times\boldsymbol{\mu}(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^2}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^5c^2}+\frac{\mathbf{n}\times\boldsymbol{\mu}(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^4c^2}\right]_{\rm ret}, \tag{26a}$$

$$\mathbf{B}_{\rm rad}=[\mathbf{n}]_{\rm ret}\times\mathbf{E}_{\rm rad}, \tag{26b}$$

are radiation fields. These fields depend on the velocity and linearly on the derivative of the acceleration as well as on the square of the acceleration. Now, in general, $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, and $\ddot{\boldsymbol{\beta}}$ are independent. This allows one to assume $\boldsymbol{\beta}=0$ and $\dot{\boldsymbol{\beta}}=0$ at least instantaneously. In this case Eqs. (26) reduce to

$$\mathbf{E}_{\rm rad}=\left[\frac{\mathbf{n}\times\boldsymbol{\mu}(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})}{Rc^2}\right]_{\rm ret}, \tag{27a}$$

$$\mathbf{B}_{\rm rad}=[\mathbf{n}]_{\rm ret}\times\mathbf{E}. \tag{27b}$$

Therefore, even when both the velocity and the acceleration of a magnetic dipole are instantaneously equal to zero (at the retarded time), the dipole can still produce a radiation field on account of the derivative of its acceleration.

VI. TOTAL RADIATED POWER BY A MOVING MAGNETIC DIPOLE

Consider now the energy flux associated with the radiation fields. It is given by the Poynting vector $\mathbf{S}=(c/4\pi)\mathbf{E}_{\rm rad}\times\mathbf{B}_{\rm rad}=(c/4\pi)|\mathbf{E}_{\rm rad}|^2\mathbf{n}$. With this vector one defines the radiated power $dN(t)/d\Omega=(\mathbf{S}\cdot\mathbf{n})R^2=(c/4\pi)|R\mathbf{E}_{\rm rad}|^2$. This is the energy per unit time and unit solid angle that is radiated in the direction $\mathbf{n}$ at time t. The radiated power $N(t)$ is connected with the radiated power $P(t')$ expressed in terms of the dipole's own time by means of the relationship [13]

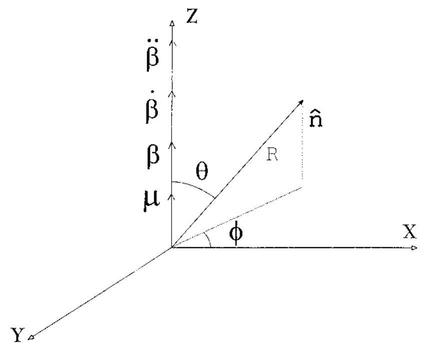

FIG. 1. Geometry of the radiation process.

$$\frac{dP(t')}{d\Omega}=\frac{dN(t)}{d\Omega}[1-\mathbf{n}\cdot\boldsymbol{\beta}]_{\rm ret}=\frac{c}{4\pi}|R\mathbf{E}|^2[1-\mathbf{n}\cdot\boldsymbol{\beta}]_{\rm ret}. \tag{28}$$

By using Eq. (26a) one obtains

$$\frac{dP(t')}{d\Omega}=\frac{1}{4\pi c^3}\left[(\mathbf{n}\times\boldsymbol{\mu})^2\left\{\frac{9(\mathbf{n}\cdot\dot{\boldsymbol{\beta}})^4}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^9}+\frac{6(\mathbf{n}\cdot\boldsymbol{\beta})^2(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^8}+\frac{(\mathbf{n}\cdot\ddot{\boldsymbol{\beta}})^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^7}\right\}\right]_{\rm ret}. \tag{29}$$

In order to find the total radiated power $P(t')$ at a fixed time t', it is necessary to specify the vectors $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, and $\ddot{\boldsymbol{\beta}}$. The simplest example of Eq. (29) is one in which the vectors $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, and $\ddot{\boldsymbol{\beta}}$ are parallel. For the sake of simplicity consider a magnetic dipole that is moving along the Z axis. The geometry is illustrated in Fig. 1. Therefore, $\boldsymbol{\beta}=\hat{\mathbf{z}}\beta$, $\dot{\boldsymbol{\beta}}=\hat{z}\dot{\beta}$, $\ddot{\boldsymbol{\beta}}=\hat{\mathbf{z}}\ddot{\beta}$, and $\boldsymbol{\mu}=\hat{\mathbf{z}}\mu$. With these specific values and with $d\Omega=\sin\theta d\theta d\phi$ and $\mathbf{n}=\hat{\mathbf{x}}(\sin\theta\cos\phi)+\hat{\mathbf{y}}(\sin\theta\sin\phi)+\hat{\mathbf{z}}\cos\theta$, Eq. (29) is first integrated over ϕ:

$$P(t')=\frac{9\mu^2}{2c^3}\left[\dot{\beta}^4\int_0^{\pi}\frac{\sin^3\theta\cos^4\theta}{(1-\beta\cos\theta)^9}d\theta\right]_{\rm ret}+\frac{3\mu^2}{c^3}\left[\dot{\beta}^2\ddot{\beta}\int_0^{\pi}\frac{\sin^3\theta\cos^3\theta}{(1-\beta\cos\theta)^8}d\theta\right]_{\rm ret}+\frac{\mu^2}{2c^3}\left[\ddot{\beta}^2\int_0^{\pi}\frac{\sin^3\theta\cos^2\theta}{(1-\beta\cos\theta)^7}d\theta\right]_{\rm ret}. \tag{30}$$

The integrals over θ are computed directly and the resulting expressions can be written in terms of powers of $\gamma=(1-\beta^2)^{-1/2}$:

$$\int_0^{\pi}\frac{\sin^3\theta\cos^4\theta}{(1-\beta\cos\theta)^9}d\theta=-\frac{4}{35}\left(\frac{21\beta^4+18\beta^2+1}{(\beta-1)^7(\beta+1)^7}\right)=\frac{4}{35}(40\gamma^{14}-60\gamma^{12}+21\gamma^{10}), \tag{31a}$$

$$\int_0^\pi \frac{\sin^3\theta\cos^3\theta}{(1-\beta\cos\theta)^8}\,d\theta=\frac{32}{105}\left(\frac{7\beta^3+3\beta}{(\beta-1)^6(\beta+1)^6}\right)$$

$$=\frac{32\beta}{105}\,(10\gamma^{12}-7\gamma^{10}),\qquad(31b)$$

$$\int_0^\pi \frac{\sin^3\theta\cos^2\theta}{(1-\beta\cos\theta)^7}\,d\theta=-\frac{4}{15}\left(\frac{7\beta^2+1}{(\beta-1)^6(\beta+1)^6}\right)$$

$$=\frac{4}{15}\,(8\gamma^{10}-7\gamma^8).\qquad(31c)$$

Substituting these values into Eq. (30) and performing some manipulation one ends up with the total power radiated by the magnetic dipole when the vectors $\boldsymbol{\mu}$, $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$, and $\ddot{\boldsymbol{\beta}}$ are parallel:

$$P(t')=\frac{18\mu^2}{35c^3}\,[\dot\beta^4(40\gamma^{14}-60\gamma^{12}+21\gamma^{10})]_{\rm ret}+\frac{32\mu^2}{35c^3}\,[\beta\dot\beta^2\ddot\beta(10\gamma^{12}-7\gamma^{10})]_{\rm ret}+\frac{2\mu^2}{15c^3}\,[\ddot\beta^2(8\gamma^{10}-7\gamma^8)]_{\rm ret}.\qquad(32)$$

Evidently, the dynamics of the dipole is disturbed by this radiation loss since it provokes a reaction force back on the dipole. To find an expression for the radiation reaction force $\mathbf{F}_{\rm rad}$, consider the low velocity limit of Eq. (32). The approximation $\beta\ll1$ implies $\gamma\approx1$ and thereby $\beta\gamma=\sqrt{\gamma^2-1}\approx0$. This approximation also implies that the effect of retardation becomes unimportant [10]. Therefore, by writing $\dot\beta=a/c$ and $\ddot\beta=\dot a/c$ Eq. (32) reduces to

$$P=\frac{18\mu^2a^4}{35c^7}+\frac{2\mu^2\dot a^2}{15c^5}.\qquad(33)$$

It should be noted that an analogous formula for the electric dipole has been recently derived [3]. As expected, the formulas for the total power radiated by electric and magnetic dipoles exhibit exactly the same structure. Therefore, the associated radiation reaction forces will exhibit necessarily the same form.

To determine $\mathbf{F}_{\rm rad}$ from the conservation of energy it is necessary that the work done by this force on the magnetic dipole in the interval $t_1<t<t_2$ must be equal to the negative of the energy radiated, that is, $\int_{t1}^{t2}\mathbf{F}_{\rm rad}\cdot\mathbf{v}dt=-\int_{t1}^{t2}Pdt$. The use of Eq. (33) yields

$$\int_{t_1}^{t_2}\mathbf{F}_{\rm rad}\cdot\mathbf{v}dt=-\frac{18\mu^2}{35c^7}\int_{t_1}^{t_2}a^4dt-\frac{2\mu^2}{15c^5}\int_{t_1}^{t_2}\dot a^2dt.\quad(34)$$

With the aid of the results

$$a^4=\frac{d}{dt}\,(a^2\mathbf{a}\cdot\mathbf{v})-3a^2\dot{\mathbf{a}}\cdot\mathbf{v},\qquad(35a)$$

$$\dot{\mathbf{a}}^2=\frac{d}{dt}\,(\mathbf{a}\cdot\dot{\mathbf{a}}-\ddot{\mathbf{a}}\cdot\mathbf{v})+\dddot{\mathbf{a}}\cdot\mathbf{v},\qquad(35b)$$

the integrals on the right-hand side of Eq. (34) can be performed by parts

$$\int_{t_1}^{t_2}\mathbf{F}_{\rm rad}\cdot\mathbf{v}dt=-\frac{18\mu^2}{35c^7}\,[a^2\mathbf{a}\cdot\mathbf{v}]_{t_1}^{t_2}-\frac{2\mu^2}{15c^5}\,[\dot{\mathbf{a}}\cdot\mathbf{a}-\ddot{\mathbf{a}}\cdot\mathbf{v}]_{t_1}^{t_2}+\frac{54\mu^2}{35c^7}\int_{t_1}^{t_2}a^2\dot{\mathbf{a}}\cdot\mathbf{v}dt-\frac{2\mu^2}{15c^5}\int_{t_1}^{t_2}\dddot{\mathbf{a}}\cdot\mathbf{v}dt,\qquad(36)$$

where $\dddot{\mathbf{a}}=\hat{\mathbf{z}}\dddot{a}$. The question now is under what conditions the first two terms in Eq. (36) vanish. A first case would be when the motion is such that $a^2\mathbf{a}\cdot\mathbf{v}=0$ and $\dot{\mathbf{a}}\cdot\mathbf{a}-\ddot{\mathbf{a}}\cdot\mathbf{v}=0$ at $t=t_1$ and $t=t_2$. A second case would be when the motion is periodic since then both quantities $a^2\mathbf{a}\cdot\mathbf{v}$ and $\dot{\mathbf{a}}\cdot\mathbf{a}-\ddot{\mathbf{a}}\cdot\mathbf{v}$ have the same value at $t=t_1$ and $t=t_2$. In a third case one might assume that the time interval t_2-t_1 is sufficiently short in such a way that the state of the system is approximately the same at $t=t_1$ and $t=t_2$ [5]. In any case one ends up with the expression

$$\int_{t_1}^{t_2}\left(\mathbf{F}_{\rm rad}-\frac{54\mu^2a^2\dot{\mathbf{a}}}{35c^7}+\frac{2\mu^2\dddot{\mathbf{a}}}{15c^5}\right)\cdot\mathbf{v}dt=0.\qquad(37)$$

From this equation one can identify the following radiation reaction force:

$$\mathbf{F}_{\rm rad}=\frac{54\mu^2}{35c^7}\,\dot{\mathbf{a}}a^2-\frac{2\mu^2}{15c^5}\,\dddot{\mathbf{a}}.\qquad(38)$$

This force is really unusual; the second of its terms is proportional to the third derivative of the acceleration, that is, to the fifth derivative of the position $\mathbf{r}(t)$. Hence, the equation of motion is a fifth order differential equation:

$$m\,\frac{d^2\mathbf{r}}{dt^2}=\mathbf{F}_{\rm ext}+\frac{54\mu^2}{35c^7}\left(\frac{d^2\mathbf{r}}{dt^2}\right)^2\frac{d^3\mathbf{r}}{dt^3}-\frac{2\mu^2}{15c^5}\,\frac{d^5\mathbf{r}}{dt^5},\qquad(39)$$

where m is the mass of the dipole and $\mathbf{F}_{\rm ext}$ is an external force. Therefore, it is necessary to specify five initial conditions in order to solve Eq. (39). When $a^2\dot{\mathbf{a}}$ and $\dddot{\mathbf{a}}$ are the same order over a brief interval, the first term in Eq. (39) is negligible when compared with the second one and thus Eq. (39) can be approximated by its linear term

$$\mathbf{F}_{\rm rad}=-\frac{2\mu^2}{15c^5}\,\dddot{\mathbf{a}}.\qquad(40)$$

Using Newton's second law this force takes the form

$$\mathbf{a}=-\tau^3\dddot{\mathbf{a}},\qquad(41)$$

where τ is a characteristic time defined by

$$\tau=\left(\frac{2\mu^2}{15mc^5}\right)^{1/3}.\qquad(42)$$

For example, the characteristic time for an electron ($\mu=9.28\times10^{-21}$ erg G^{-1} and $m=9.11\times10^{-28}$ G) is $\tau=8.05\times10^{-23}$ sec. It is interesting to note that this characteristic time is one order of magnitude greater than the characteristic time involved in the usual Abraham-Lorentz formula associated to the charge of the electron, $\mathbf{a}=\tau\dot{\mathbf{a}}$, which has the value $\tau=6.26\times10^{-24}$ sec.

The three linearly independent solutions of Eq. (41) are

$$\mathbf{a}_1(t)=\mathbf{k}_1 e^{-t/\tau}, \tag{43a}$$

$$\mathbf{a}_2(t)=\mathbf{k}_2 e^{t/2\tau}\cos(\sqrt{3}t/2\tau), \tag{43b}$$

$$\mathbf{a}_3(t)=\mathbf{k}_3 e^{t/2\tau}\sin(\sqrt{3}t/2\tau), \tag{43c}$$

where $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ are vectorial constants. Evidently, $\mathbf{a}_2$ and $\mathbf{a}_3$ are *runaway solutions*. However, $\mathbf{a}_1$ is a reasonable solution from a physical point of view since it predicts that acceleration spontaneously decreases exponentially with time, which clearly agrees with the law of inertia. This prediction is contrary to that of the usual Abraham-Lorentz equation of the point charge which has no *natural* solution consistent with the law of inertia.

Finally, it is interesting to note that Eq. (40) corresponds (up to a constant) with the following expression derived by Smirnov [7]:

$$\mathbf{F}_{\text{rad}}=-\frac{2\mu^2}{3c^5}\dddot{\mathbf{a}}, \tag{44}$$

for a quantum nonrelativistic particle with zero electric charge, mass m, and spin magnetic moment $\boldsymbol{\mu}=g\mu_0\boldsymbol{\sigma}$ [here g is the g factor of the particle, μ_0 is the corresponding magneton, and $\boldsymbol{\sigma}=\{\sigma_i\}$ $(i=1,2,3)$ is the set of Pauli matrices].

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APPENDIX: DERIVATION OF EQS. (4), (8), AND (9)

Consider the ith Cartesian component of the integrand of the left-hand side of Eq. (4):

$$\left(\frac{\delta(u)\nabla'\times\mathbf{M}}{Rc}\right)^i=\frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}, \tag{A1}$$

where ε^{ipq} is the three-dimensional Levi-Civita symbol with $\varepsilon^{123}=1$; $M^i=(\mathbf{M})^i$; $(\nabla')_i=\partial/\partial x'^i$, and $R=|x^i-x'^i|$. The summation convention on repeated indices is adopted. The right-hand side of Eq. (A1) can be expressed as

$$\frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}=\frac{\partial}{\partial x'^p}\left(\frac{\varepsilon^{ipq}\delta(u)M_q}{Rc}\right)-\frac{\varepsilon^{ipq}M_q}{Rc}\frac{\partial\delta(u)}{\partial x'^p}-\frac{\varepsilon^{ipq}\delta(u)M_q}{c}\frac{\partial}{\partial x'^p}\left(\frac{1}{R}\right). \tag{A2}$$

The derivatives in the last two terms are

$$\frac{\partial\delta(u)}{\partial x'^p}=-\frac{n_p}{c}\frac{\partial\delta(u)}{\partial t'},\quad \frac{\partial}{\partial x'^p}\left(\frac{1}{R}\right)=\frac{n_p}{R^2}, \tag{A3}$$

where $n_p=(\mathbf{R})_p/R$. With these derivatives, Eq. (A2) takes the form

$$\frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}=\frac{\partial}{\partial x'^p}\left(\frac{\varepsilon^{ipq}\delta(u)M_q}{Rc}\right)-\frac{\delta(u)\varepsilon^{ipq}n_pM_q}{R^2c}+\frac{\varepsilon^{ipq}n_pM_q}{Rc^2}\frac{\partial\delta(u)}{\partial t'}. \tag{A4}$$

The last term may be rewritten as

$$\frac{\varepsilon^{ipq}n_pM_q}{Rc^2}\frac{\partial\delta(u)}{\partial t'}=\frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}n_pM_q}{Rc^2}\right)-\frac{\delta(u)\varepsilon^{ipq}n_p}{Rc^2}\frac{\partial M_q}{\partial t'}, \tag{A5}$$

and therefore Eq. (A4) takes the final form

$$\frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}=-\delta(u)\left(\frac{\varepsilon^{ipq}n_pM_q}{R^2c}+\frac{\varepsilon^{ipq}n_p}{Rc^2}\frac{\partial M_q}{\partial t'}\right)+\frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}n_pM_q}{Rc^2}\right)+\frac{\partial}{\partial x'^p}\left(\frac{\varepsilon^{ipq}\delta(u)M_q}{Rc}\right). \tag{A6}$$

When this expression is integrated over space and time one obtains

$$\int\int\frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}d^3x'dt'=-\int\int\delta(u)\left(\frac{\varepsilon^{ipq}n_pM_q}{R^2c}+\frac{\varepsilon^{ipq}n_p}{Rc^2}\frac{\partial M_q}{\partial t'}\right)d^3x'dt'+\int\left[\int\frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}n_pM_q}{Rc^2}\right)dt'\right]d^3x'+\int\left[\int\frac{\partial}{\partial x'^p}\left(\frac{\varepsilon^{ipq}\delta(u)M_q}{Rc}\right)d^3x'\right]dt'. \tag{A7}$$

The time integration in the second term of the right-hand side of Eq. (A7) gives zero because the delta function vanishes for $t'=\pm\infty$. The volume integral of the last term of Eq. (A7) becomes a surface integral, and hence vanishes at infinity assuming the magnetization is contained in a finite volume. Thus, Eq. (A7) reduces to

$$\int\int \frac{\varepsilon^{ipq}\delta(u)}{Rc}\frac{\partial M_q}{\partial x'^p}d^3x'dt' = -\int\int \delta(u)\left(\frac{\varepsilon^{ipq}n_pM_q}{R^2c}+\frac{\varepsilon^{ipq}n_p}{Rc^2}\frac{\partial M_q}{\partial t'}\right)d^3x'dt'. \tag{A8}$$

The vector version of this expression is precisely Eq. (4).

Equation (8) will be now demonstrated. The ith Cartesian component of the integrand of the left-hand side of Eq. (8),

$$\left(\frac{\delta(u)\nabla'\times(\nabla'\times\mathbf{M})}{Rc}\right)^i = \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s}. \tag{A9}$$

The right-hand side of this equation can be expressed as

$$\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s} = -\frac{\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\frac{\partial\delta(u)}{\partial x'_s} - \delta(u)\varepsilon^{ipq}\varepsilon_{qst}M^t\frac{\partial}{\partial x'_s}\left(\frac{n_p}{R^2}\right) + \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\right). \tag{A10}$$

The derivatives in the first two terms of the right-hand side are

$$\frac{\partial\delta(u)}{\partial x'_s} = -\frac{n^s}{c}\frac{\partial\delta(u)}{\partial t'},$$

$$\frac{\partial}{\partial x'_s}\left(\frac{n_p}{R^2}\right) = \frac{3n_pn^s-\delta_p^s}{R^3} = \frac{4\pi}{3}\delta_p^s\delta(\mathbf{x}-\mathbf{x}'). \tag{A11}$$

With these derivatives, Eq. (A10) becomes

$$\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s} = \frac{\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\frac{\partial\delta(u)}{\partial t'} - \delta(u)\varepsilon^{ipq}\varepsilon_{qst}M^t\left(\frac{3n_pn^s-\delta_p^s}{R^3}\right) + \frac{4\pi}{3}\delta(u)\varepsilon^{ipq}\varepsilon_{qst}M^t\delta_p^s\delta(\mathbf{x}-\mathbf{x}') + \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\right). \tag{A12}$$

The first term of the right-hand side can be written as

$$\frac{\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\frac{\partial\delta(u)}{\partial t'} = \frac{\partial}{\partial t'}\left(\frac{\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\right) - \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^s}{R^2c}\frac{\partial M^t}{\partial t'} \tag{A13}$$

and therefore Eq. (A12) takes the form

$$\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s} = -\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^s}{R^2c}\frac{\partial M^t}{\partial t'} - \delta(u)\varepsilon^{ipq}\varepsilon_{qst}M^t\left(\frac{3n_pn^s-\delta_p^s}{R^3}\right) + \frac{4\pi}{3}\delta(u)\varepsilon^{ipq}\varepsilon_{qst}M^t\delta_p^s\delta(\mathbf{x}-\mathbf{x}') + \frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\right) + \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\right). \tag{A14}$$

Using the identity $\varepsilon^{ipq}\varepsilon_{qst} = \varepsilon^{ipq}\varepsilon_{stq} = \delta_s^i\delta_t^p - \delta_t^i\delta_s^p$ in the first three terms of the right-hand side of Eq. (A14) it becomes

$$\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s} = -\delta(u)\left(\frac{3n^in_tM^t-M^i}{R^3}+\frac{n^in_t\dot{M}^t-\dot{M}^i}{R^2c}\right) - \frac{8\pi}{3}\delta(u)M^i\delta(\mathbf{x}-\mathbf{x}') + \frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\right) + \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\right). \tag{A15}$$

Integration over space and time of this expression yields

$$\int\int \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s}d^3x'dt' = -\int\int \delta(u)\left(\frac{3n^in_tM^t-M^i}{R^3}+\frac{n^in_t\dot{M}^t-\dot{M}^i}{R^2c}\right)d^3x'dt' - \frac{8\pi}{3}\int\int \delta(u)M^i\delta(\mathbf{x}-\mathbf{x}')d^3x'dt' + \int\left[\int \frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{R^2c}\right)dt'\right]d^3x' + \int\left[\int \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{R^2}\right)d^3x'\right]dt'. \tag{A16}$$

The time integration in the second term of the right-hand side of Eq. (A16) gives zero because the delta function vanishes for $t' = \pm\infty$. The volume integral of the last term of Eq. (A16) becomes a surface integral, and hence vanishes at infinity assuming the magnetization is confined to a finite region of space. Thus, Eq. (A16) reduces to

$$\int\int \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{R^2}\frac{\partial M^t}{\partial x'_s}\,d^3x'\,dt' = -\int\int \delta(u)\left(\frac{3n^in_tM^t - M^i}{R^3} + \frac{n^in_t\dot{M}^t - \dot{M}^i}{R^2c}\right)d^3x'\,dt' - \frac{8\pi}{3}M^i. \tag{A17}$$

When this expression is written in vector notation Eq. (8) is obtained.

The derivation of Eq. (9) is similar to that of Eq. (8). Accordingly, after a manipulation similar to that for obtaining Eq. (A14), the following identity is derived:

$$\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{Rc}\frac{\partial M^t}{\partial x'_s} = -\delta(u)\left(\frac{2n^in_tM^t}{R^3} + \frac{n^in_t\dot{M}^t - \dot{M}^i}{Rc^2}\right) + \frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{Rc^2}\right) + \frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{Rc}\right). \tag{A18}$$

If this identity is integrated over space and time the result is

$$\int\int \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{Rc}\frac{\partial M^t}{\partial x'_s}\,d^3x'\,dt' = -\int\int \delta(u)\left(\frac{2n^in_tM^t}{R^3} + \frac{n^in_t\dot{M}^t - \dot{M}^i}{Rc^2}\right)d^3x'\,dt' + \int\left(\frac{\partial}{\partial t'}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pn^sM^t}{Rc^2}\right)dt'\right)d^3x' + \int\left(\frac{\partial}{\partial x'_s}\left(\frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_pM^t}{Rc}\right)d^3x'\right)dt'. \tag{A19}$$

The time integration in the second term of the right-hand side of Eq. (A19) gives zero because the delta function vanishes for $t' = \pm\infty$. The volume integral of the last term of Eq. (A19) becomes a surface integral, and hence vanishes at infinity assuming the magnetization is confined to a finite region of space. Thus, Eq. (A19) reduces to

$$\int\int \frac{\delta(u)\varepsilon^{ipq}\varepsilon_{qst}n_p}{Rc}\frac{\partial M^t}{\partial x'_s}\,d^3x'\,dt' = -\int\int \delta(u)\left(\frac{2n^in_tM^t}{R^3} + \frac{n^in_t\dot{M}^t - \dot{M}^i}{Rc^2}\right)d^3x'\,dt'. \tag{A20}$$

The vector version of this equation is precisely Eq. (9).

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